MATH 2050C Lecture on 4/18/2020

Reminder: PS 9 due this Friday, PS 10 posted and due Apr 20 (Monday).

Midterm available on BLACKBOARD at 8:30am, Apr 15. Submission deadline: 8:30am, Apr 16.

NO CLASS THIS FRIDAY (Easter) & NEXT WEDNESDAY (Midterm)

Tips for Midterm: mainly focus on Ch. 2 & 3.

Recall: $f: A \to \mathbb{R}$ is cts at $c \in A$ iff

$\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ s.t. $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$.

first given — chosen next — chosen next

FACT: • c NOT cluster pt. $\Rightarrow$ f is automatically cts at c

• c cluster pt. $\Rightarrow$ "f is cts at c" $\Leftrightarrow$ "$\lim_{x \to c} f(x) = f(c)$" exists & $= f(c)$

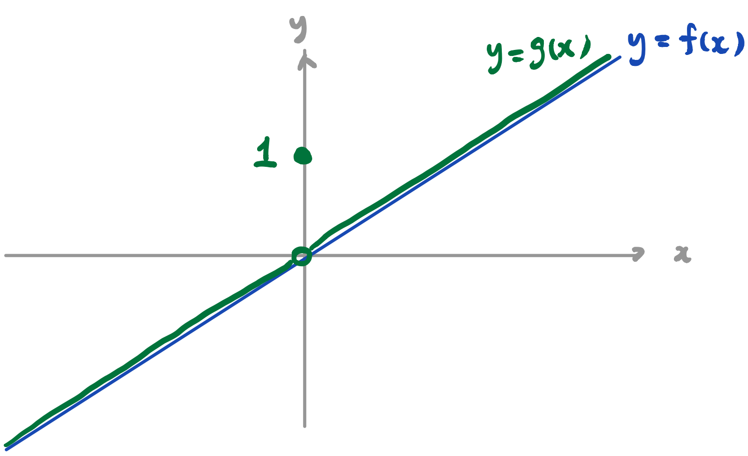
By seq. criteria, $\Leftrightarrow$ $\forall$ seq $(x_n) \to c$, $\lim_{n \to \infty} f(x_n) = f(c)$.

Caution: Continuity of f at c is sensitive to the value $f(c)$

while $\lim_{x \to c} f(x)$ does NOT.

Example 1: Consider the functions $f, g: \mathbb{R} \to \mathbb{R}$

$$f(x) := \begin{cases} x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \qquad g(x) := \begin{cases} x, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$



Note: f is cts everywhere

But g is NOT cts at 0.

Pf: Consider $(\frac{1}{n}) \to 0$.

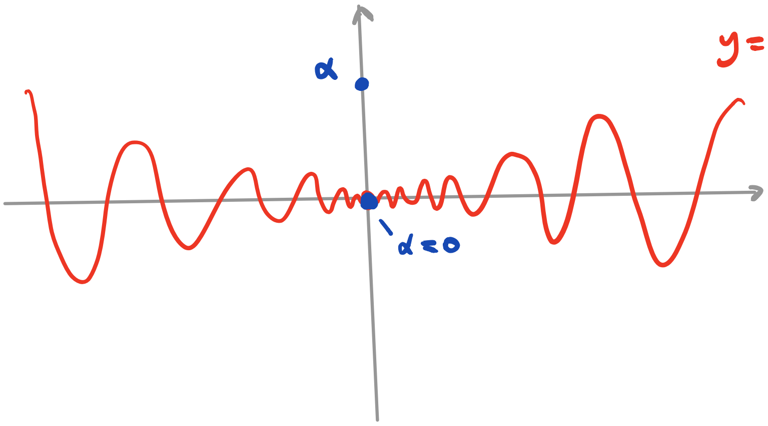
$\lim_{n \to \infty} g(\frac{1}{n}) = \lim_{n \to \infty} \frac{1}{n} = 0 \neq g(0)$

$\blacksquare$

Example 2: Find $\alpha \in \mathbb{R}$ s.t. the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} x \sin \frac{1}{x} & , \text{ if } x \neq 0 \\ \alpha & , \text{ if } x = 0 \end{cases}$$

is cts everywhere.



Pf: f is cts at 0 $\iff$

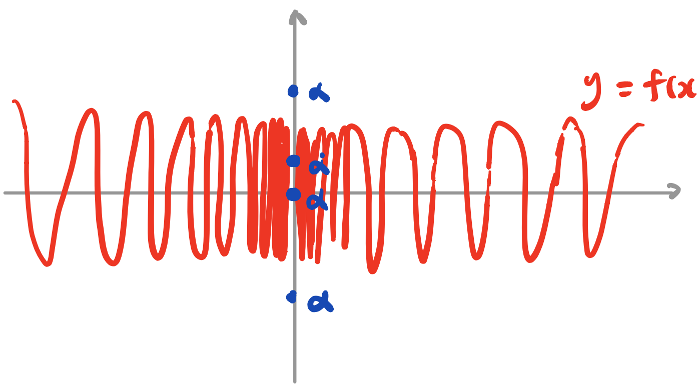
$$\alpha = f(0) = \lim_{x\to 0} f(x)$$

$$= \lim_{x \to 0} x \sin \frac{1}{x} = 0$$

Example 3: Find $\alpha \in \mathbb{R}$ s.t. the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} \sin \frac{1}{x} & , \text{ if } x \neq 0 \\ \alpha & , \text{ if } x = 0 \end{cases}$$

is cts everywhere.



Note: Such an α does NOT exist.

$\because$ $\lim_{x \to 0} f(x)$ does NOT exist.

Q: How to construct NEW cts functions from OLD ones?

Thm: Let $f, g : A \to \mathbb{R}$ be cts (at $c \in A$).

Then, the functions $f \pm g$, fg, f/g, $|f|$, $\sqrt{f}$ are cts (at $c \in A$)

wherever they are defined.

Proof: By limit thms for functions.

Thm: (Composition of functions)

Let $f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$ be functions s.t. $f(A) \subseteq B$,

(i) f is cts at $c \in A$

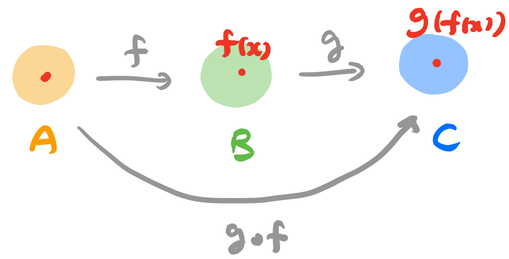
(ii) g is cts at $b := f(c) \in B$

Then, $g \circ f$ is cts at $c \in A$.

Recall: $f: A \to \mathbb{R}$
$g: B \to \mathbb{R}$
and $f(A) \subseteq B$

Define $g \circ f: A \to \mathbb{R}$

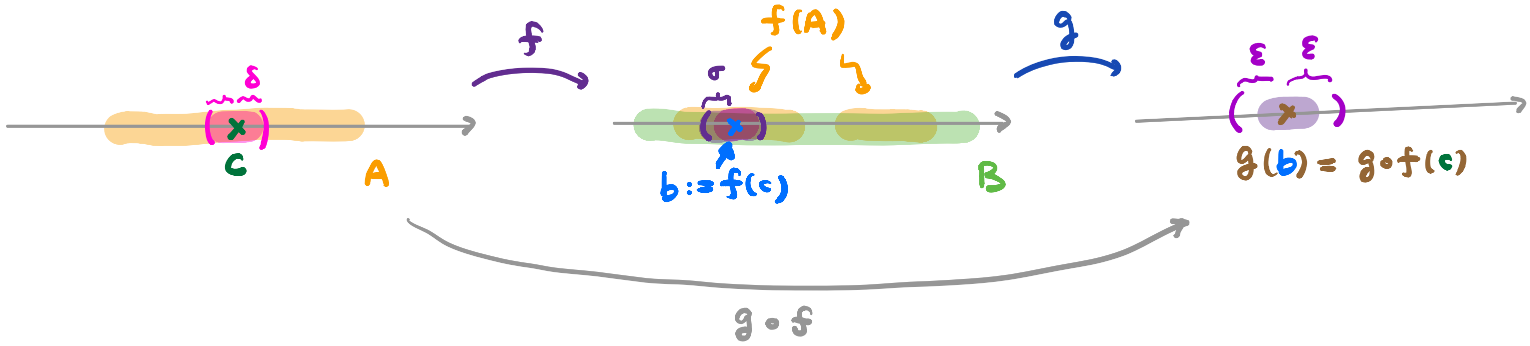
$g \circ f(x) := g(f(x))$



Proof:

Want to show: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$|g \circ f(x) - g \circ f(c)| < \varepsilon \quad \text{whenever} \quad |x - c| < \delta.$$



Let $\varepsilon > 0$.

- g is cts at b $\Rightarrow$ $\exists\, \sigma = \sigma(\varepsilon) > 0$ s.t.

$$|g(y) - g(b)| < \varepsilon \quad \text{whenever} \quad |y - b| < \sigma \qquad (*)$$

- f is cts at c $\Rightarrow$ $\exists\, \delta = \delta(\sigma) > 0$ s.t.

(treating $\sigma = \varepsilon$ in ε-δ def^{n})

$$|f(x) - f(c)| < \sigma \quad \text{whenever} \quad |x - c| < \delta \qquad (**)$$

So, for any $x \in A$ s.t. $|x - c| < \delta$,

$$|g \circ f(x) - g \circ f(c)| = |g(f(x)) - g(\underbrace{f(c)}_{=b})| \underset{(**)}{\overset{(*)}{<}} \varepsilon.$$

$\blacksquare$